Weekly Focus: probability circle graphs
Weekly Skill: compute permutations and combinations, solve graph word problems

LESSON 15: Probability and Circle Graphs

Lesson Summary: In the warm up, students will solve a probability problem. In Activity 1, they will review notes and examples of probability that were presented in the previous lesson. In Activity 2, they learn about and do permutations. In Activity 3, they will practice combinations. In Activity 4, they will look at examples of circle graphs. In Activity 5, they will solve problems using circle graphs. In Activity 6, they will make their own circle graphs. There are also an exit ticket and extra problem at the end. Estimated time for the lesson is two hours.

Materials Needed for Lesson 15:

- 2 worksheets (15.1 and 15.2) with answers on permutations and combinations (attached)
- Handout 15.3 for students with notes on factorials, permutations, and combinations (attached)
- *Mathematical Reasoning Test Preparation for the 2014 GED Test* Student Book (pages 36-37) and Workbook (pages 50-53) by Steck-Vaughn © 2013.
- Paper, rulers, and markers
- Exit ticket (attached)

- Teacher Note: These probability skills are needed for the GED test, but they are not included in the core text. Use the attached notes as the text for this lesson.
- Teacher Note: There is a fun application activity using QR codes. It is available at the link below but is not included here due to lack of time in this lesson: http://www.yummymath.com/wp-content/uploads/QR_codes.pdf

Objectives: Students will be able to:

- Understand and compute factorials and permutations
- Understand and compute combination problems
- Answer questions using circle graphs
- Make a circle graph

ACES Skills Addressed: N, CT, LS
CCRS Mathematical Practices Addressed: Reason Abstractly and Quantitatively, Use Appropriate Tools Strategically, Mathematical Fluency
Levels of Knowing Math Addressed: Intuitive, Abstract, and Application

Notes:

You can add more examples if you feel students need them before they work. Any ideas that concretely relates to their lives make good examples.

For more practice as a class, feel free to choose some of the easier problems from the worksheets to do together. The "easier" problems are not necessarily at the beginning of each worksheet. Also, you may decide to have students complete only part of the worksheets in class and assign the rest as homework or extra practice.

The GED Math test is 115 minutes long and includes approximately 46 questions. The questions have a focus on quantitative problem solving (45%) and algebraic problem solving (55%).

Students must be able to understand math concepts and apply them to new situations, use logical reasoning to explain their answers, evaluate and further the reasoning of others, represent real world problems algebraically and visually, and manipulate and solve algebraic expressions.

This computer-based test includes questions that may be multiple-choice, fill-in-the-blank, choose from a drop-down menu, or drag-and-drop the response from one place to another.

The purpose of the GED test is to provide students with the skills necessary to either further their education or be ready for the demands of today's careers.

Lesson 15 Warm-up: Solve the probability question	**Time: 5 Minutes**

Write on the board: Nora is planning a 10-day vacation. She wants a different outfit to wear every day but will wear the same clothes more than one day.

Basic Questions:

- If she brings 2 pairs of pants, how many shirts does she need? *(5 shirts because 2 x 5 = 10)*
- How many days would she wear each pair of pants? *(5 days)* shirt? *(2 days)*

Extension Questions:

- If she extends her vacation to 15 days and brings a pair of shorts, does she need any more shirts? *(No because 3 pants x 5 shirts= 15 outfits)*
- Now how many days would she wear each shirt? *(3 days)*

Lesson 15 Activity 1: Review Fundamental Counting Principle	**Time: 10 Minutes**

1) Give the students the pages of notes attached to this lesson.
2) Review the fundamental counting principle that they learned in the previous lesson by going through **pages 1 and 2 of these notes** with examples.
3) The warm up activity for this lesson used the same principle.

Lesson 15 Activity 2: Factorials and Permutations	**Time: 15 Minutes**

1. Example A: Have 3 students volunteer to come to the front of the class and stand in line. Ask the class: If we are to change the order of who is in line 1st, 2nd, 3rd every time, how many possible combinations can we make? Let the students work it out.
   a. The answer will be 6 (like the example on arranging the letters ACT on p.3-4 of notes)
2. Example B: Nick is making a playlist of songs he likes. Right now his list has only 4 songs on it. How many ways can he arrange the list?
   a. The answer is 4x3x2x1=24. The reason is for the first song there are 4 choices, for the 2nd

song only 3 choices remain, for the 3rd song only 2 choices remain, and there is one song left at the end.

3. Do you notice a pattern? Yes, we multiply the total number we are given by subsequent numbers in descending order. This is called a factorial and is written as n!

4. Do Worksheet 15.1. These are simple permutations because we are including all items when counting. We just need to do the factorial n! **Permutations mean that the order of the responses matter and you do not repeat any item.**

5. Now look at notes and examples on pages 3 and 4. In these permutations, not all items are needed in the answer, and that is the purpose for the **permutation formula**. (The example on page 4 shows how we choose 3 people out of a total of 7 people.)

Lesson 15 Activity 3: Combinations	Time: 25 Minutes

1) Sometimes the order of the items doesn't matter. **Combinations are used when there still is no repetition but the order does not matter.**

2) Look at the license plate example (bottom of page 5 of notes).

3) Read the other examples pages 6 to 8.

4) Do **Worksheet 15.2** on combinations. Go over challenging problems.

Worksheet 15.1—Easy Permutation (Use factorials only)

(1) Sean assembled 4 toy train cars on a track. How many different ways could he have ordered the cars?

(2) How many 6-digit numbers can be created from the digits 8, 4, 7, 6, 1 and 5 without repeating any?

(3) How many different ways can Andrew, Madeline, Jack and Jordan stand in a row?

(4) In a 6-person race, how many different ways can the runners arrive at the finish line?

(5) Each person in a 4-person committee is assigned a different job. How many ways can the jobs be assigned?

(6) How many ways can the letters H, O, W, R and Q be ordered without repeating any?

(7) 6 people walk into a fast-food restaurant at the same time. How many different ways can they stand in line?

(8) Madison was asked to hang 7 student paintings in a row on the wall. How many different ways could she arrange them?

Worksheet 15.1—Easy Permutation Answers

(1) Sean assembled 4 toy train cars on a track. How many different ways could he have ordered the cars?

$nPk = 4! = 24$

(2) How many 6-digit numbers can be created from the digits 8, 4, 7, 6, 1 and 5 without repeating any?

$nPk = 6! = 720$

(3) How many different ways can Andrew, Madeline, Jack and Jordan stand in a row?

$nPk = 4! = 24$

(4) In a 6-person race, how many different ways can the runners arrive at the finish line?

$nPk = 6! = 720$

(5) Each person in a 4-person committee is assigned a different job. How many ways can the jobs be assigned?

$nPk = 4! = 24$

(6) How many ways can the letters H, O, W, R and Q be ordered without repeating any?

$nPk = 5! = 120$

(7) 6 people walk into a fast-food restaurant at the same time. How many different ways can they stand in line?

$nPk = 6! = 720$

(8) Madison was asked to hang 7 student paintings in a row on the wall. How many different ways could she arrange them?

$nPk = 7! = 5{,}040$

Worksheet 15.2—Combinations

(1) How many different 3-person teams can be created from a classroom of 11 students?

(2) Kayla asked the pet store owner for any 2 baby mice from a cage containing 8. How many possible combinations of mice could be picked?

(3) There are 8 different marbles in a jar. How many different sets could you get by randomly picking 3 of them from the jar?

(4) 7 wild shoppers are rushing to grab one of the last 4 Elmo toys. How many different sets of shoppers could come away with a toy?

(5) A painter was carrying 6 pails of different colored paint and dropped 3 of them, making a big mess. How many combinations of colors could he have spilled?

(6) 4 names will be picked from a jar to win prizes. There are a total of 9 names in the jar. How many different combinations of names can be picked?

Worksheet 15.2—Combinations Answers

(1) How many different 3-person teams can be created from a classroom of 11 students?

$$nCk = \frac{11!}{3!(11-3)!} = 165$$

(2) Kayla asked the pet store owner for any 2 baby mice from a cage containing 8. How many possible combinations of mice could be picked?

$$nCk = \frac{8!}{2!(8-2)!} = 28$$

(3) There are 8 different marbles in a jar. How many different sets could you get by randomly picking 3 of them from the jar?

$$nCk = \frac{8!}{3!(8-3)!} = 56$$

(4) 7 wild shoppers are rushing to grab one of the last 4 Elmo toys. How many different sets of shoppers could come away with a toy?

$$nCk = \frac{7!}{4!(7-4)!} = 35$$

(5) A painter was carrying 6 pails of different colored paint and dropped 3 of them, making a big mess. How many combinations of colors could he have spilled?

$$nCk = \frac{6!}{3!(6-3)!} = 20$$

(6) 4 names will be picked from a jar to win prizes. There are a total of 9 names in the jar. How many different combinations of names can be picked?

$$nCk = \frac{9!}{4!(9-4)!} = 126$$

Handout 15.3—Counting Methods: Notes and Examples (9 pages)

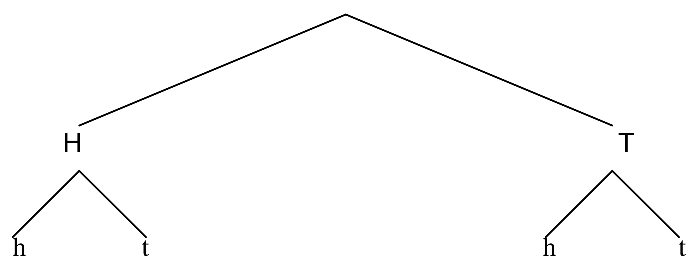
COUNTING METHODS

From our preliminary work in probability, we often found ourselves wondering how many different scenarios there were in a given situation. In the beginning of that chapter, we merely tried to list all the possible outcomes, hoping we didn't miss any. When we determined that was not good enough, we began to use a tree diagram to lend some order to the listing. While that worked out well for smaller sample spaces, we quickly saw its limitations when there were a great number of outcomes. For that reason, we will look at some counting methods that should make our work a lot easier

Methods Used for Counting

1. Listing
2. Cartesian product
3. Tree Diagram
4. Fundamental Counting Principle
5. Permutation
6. Combination

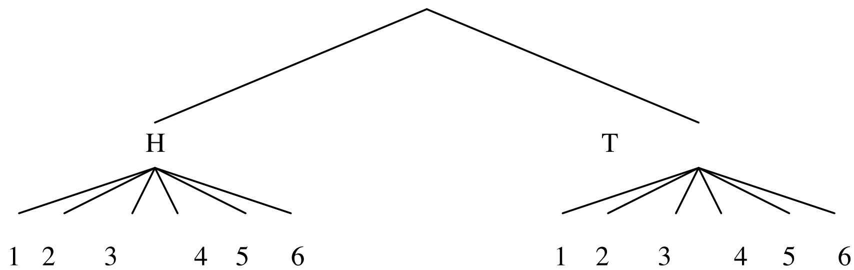
If we looked at the number of outcomes in a sample space being described using a tree diagram, we might notice a pattern that would suggest a counting method. For instance, if I drew the tree diagram for tossing 2 coins, I would see there would be four possible outcomes – Hh, Ht, Th, and Tt.



With a little investigation, I might also notice there were two possible outcomes throwing the first coin and two possible outcomes tossing the second.

If I looked at another example, say throwing a coin and rolling a die. Drawing a tree diagram I would quickly see there are twelve possible outcomes – H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, and T6. Again, you might notice there are two possible outcomes when tossing the coin and six outcomes when rolling the die.

Handout 15.3 Page 2



By doing a few more of these problems, I might begin to see a pattern that would suggest a way of determining the total number of outcomes without listing and without using a tree diagram.

In the first example, we tossed two coins and discovered there were four possible outcomes. Each stage of the experiment having two outcomes, we notice that $2x2=4$.

In the second example, tossing a coin and rolling a die, we discovered using a tree diagram there were twelve possible outcomes. By looking at each stage of the experiment, we see that there are two outcomes possible for stage one, and rolling the die resulted in six possible outcomes. Notice that by multiplying 2 X 6 we end up with 12 possible outcomes. Getting excited?

That excitement will lead us to the Fundamental Counting Principle.

Fundamental Counting Principle – In general, if there are ***m*** choices for doing one thing, and after that occurs, there are ***n*** choices for doing another, then together they can be done in ***m x n*** ways.

This counting principle will allow me to determine how many different outcomes exist quickly in my head that could be verified using tree diagrams.

In the coin tossing example, since there were 2 things that could happen on the first toss, followed by two things that could happen on the second toss, the Fundamental Counting Principle states that there will be 2x2 or 4 possible outcomes.

Example
Suppose Jennifer has three blouses, two pairs of slacks, and four pairs of shoes. Assuming no matter what she wears, they all match, how many outfits does she have altogether?

She has three choices for a blouse, two choices for her slacks, and four choices for her shows. Using the Fundamental Counting Principle, she has 3 x 2 x 4 or 24 different outfits.

Example
Abe, Ben, and Carl are running a race, in how many ways can they finish?

I could draw a tree diagram to see all the possible outcomes or I could use the Fundamental Counting Principle. There are three ways I could choose the winner, and after that occurs, there are two ways to pick second place, and one way to pick the third place finisher. Therefore there are 3 x 2 x 1 or 6 different way these three boys could finish the race.

Handout 15.3 Page 3

Before we go on, we need to learn a little notation.

Factorials

In the last example, we saw that we had to multiply 3 x 2 x 1. As it turns out, we will have a number of opportunities to multiply numbers like 5 x 4 x 3 x 2 x 1. A difficulty with this is that if I have to multiply 30 x 29 x 28 x …x 3 x 2 x 1, that is a lot of writing and a lot of space.

So we are going to abbreviate products that start at one number and work their way back to one by using an exclamation point (!). In math, however, that won't mean the number is excited. And we won't call it an exclamation point, we'll call it a factorial. So 5! Is read as five factorial.

5!	=	5 x 4 x 3 x 2 x 1
	=	120
4!	=	4 x 3 x 2 x 1
	=	24
3!	=	3 x 2 x 1
	=	6
2!	=	2 x 1
	=	2
1!	=	1

Later on, we will find it to our advantage to define 0! = 1.

Now that we have learned factorials, we can more efficiently determine the size of sample spaces.

One of the best strategies to use when trying to find out how many different outcomes are in a sample space is using the Fundamental Counting Principle. Find out how many ways the first event can occur, then multiply that by the number of ways events can occur in subsequent events.

Example
How many outcomes are there if two dice were thrown?

There are 6 outcomes on the first die, and after that occurs, there are 6 outcomes on the second die.

Using the Fundamental Counting Principle, there are 6 x 6 or 36 possible outcomes.

Example
How many different ways can the letters in the word "ACT" be arranged?

Handout 15.3 Page 4

There are six different ways to write those letters. We can see that in the following list.

ACT
CTA
TAC
CAT
ATC
TCA

We could have determined there were six by using the Fundamental Counting Principle, three ways to pick the first letter, 2 ways to pick the second, then one way to pick the third. That could also have been described using 3!

Let's look at another example.

Example
If a different person must be selected for each position, in how many ways can we choose the president, vice president, and secretary from a group of seven members if the first person chosen is the president, the second the vice president, and the third is the secretary?

We have a total of 7 people taken three at a time. Using the Fundamental Counting Principle, the first person can be chosen 7 ways, the next 6, and the third 5, we have
7 x 6 x 5 or 210 ways of choosing the officers.

Another way of doing the same problem is by developing a formula. Let's see what that might look like and define it.

Permutation is an arrangement of objects in which the order matters – without repetition.

Order typically matters when there is position or awards. Like first place, second place, or when someone is named president, or vice president. Various notations are used to represent the number of permutations of a set of **n** objects taken **r** at a time nPr and $P(n,r)$ are the most popular.

In the last example, we would use the notation ${}_7P_3$ to represent picking three people out of the seven. We could then use the Fundamental Counting principle to determine the number of permutations.

$$
\begin{aligned}
{}_7P_3 &= 7x6x5 \\
&= 210
\end{aligned}
$$

Generalizing this algebraically, we could develop the following formula for a permutation.

$$n\mathrm{Pr} = \frac{n!}{(n-r)!}$$

Using that formula for the last example would give us

Handout 15.3 Page 5

$$ {}_7P_3 = \frac{7!}{(7-3)!} $$

$$ = \frac{7!}{4!} $$

$$ = \frac{7x6x5x\cancel{4}x\cancel{3}x\cancel{2}x\cancel{1}}{\cancel{4}x\cancel{3}x\cancel{2}x\cancel{1}} $$

$= 7x6x5$ or 210

Example
Three friends buy an all day pass to ride a two seated bike, if only two of them can ride at a time, how many possible seating arrangements are there?

Since order matters (sitting in front or back), using the Fundamental Counting Principle, the number of permutations of 3 friends taken 2 at a time is 3 x 2 or 6.

The front seat can be chosen three ways, and after that occurs, the person in the back seat can be chosen two ways. We could have used the formula.

$$ {}_3P_2 = \frac{3!}{(3-2)!} $$

Example
There are 5 runners in a race. How many different permutations are possible for the places in which the runners finish?

By formula, we have a permutation of 5 runners being taken 5 at a time.

$$ {}_5P_5 = \frac{5!}{(5-5)!} $$

$$ = \frac{5!}{0!} $$

= 5! Or 5 x 4 x 3 x 2 x 1

Notice, we could have just as easily used the Fundamental Counting Principle to solve this problem.

Using a permutation or the Fundamental Counting Principle, order matters. A permutation does not allow repetition. For instance, in finding the number of arrangements of license plates, the digits can be re-used. In other words, someone might have the license plate

Handout 15.3 Page 6

333 333. To determine the possible number of license plates, I could not use the permutation formula because of the repetitions, I would have to use the Fundamental Counting Principle.

Since there are 10 ways to chose each digit on the license plate, the number of plates would be determined by – 10 x 10 x 10 x 10 x 10 x 10 or 1,000,000

Combination is an arrangement of objects in which the order does not matter – without repetition.

This is different from a permutation because the order does not matter. If you change the order, you don't change the group, you do not make a new combination.

So, a dime, nickel, and penny is the same combination of coins as a penny, dime, and nickel.

Example
Bob has four golf shirts. He wants to take two of them on his golf outing. How many different combinations of two shirts can he take?

Using the Fundamental Counting Principle, Bob could choose his first shirt four ways, his second shirt three ways – 12 ways.

But, hold on a minute. Let's say those shirts each had a different color, by using the Fundamental Counting Principle, that would suggest taking picking the blue shirt, then the yellow is different from picking the yellow first, then the blue.

We don't want that to happen since the order does not matter. To find the number of combinations, in other words, eliminating the order of the two shirts, we would divide the 12 permutations by 2! Or 2 x 1

There would be six different shirt combinations Bob could take on his outing.

In essence, a combination is nothing more than a permutation that is being divided by the different orderings of that permutation.

The notation we will use will follow that of a permutation, either $nCr \text{ or } C(n,r)$

$$nCr = \frac{n\text{Pr}}{r!}$$

Handout 15.3 Page 7

Simplifying that algebraically, we have

$$nCr = \frac{n!}{(n-r)!r!}$$

In a permutation, A,B is different from B,A because order is important. In a combination, you would either have A,B or B,A – not both, they are the same grouping.

Example
From among 12 students trying out for the basketball team, how many ways can 7 students be selected?

Does the order matter? Is this a permutation or combination? Well, if you were going out for the team and a list was printed, would it matter if you were listed first or last? All you would care about is that your name is on the list. The order is not important, therefore this would be a combination problem of 12 students take 7 at a time.

$$_{12}C_7 = \frac{12!}{(12-7)!7!}$$

$$= \frac{12!}{5!7!}$$

$$= \frac{12\ 11\ 10\ 9\ 8\ \cancel{7}\ \cancel{6}\ \cancel{5}\ \cancel{4}\ \cancel{3}\ \cancel{2}\ \cancel{1}}{5\ 4\ 3\ 2\ 1\ \cancel{7}\ \cancel{6}\ \cancel{5}\ \cancel{3}\ \cancel{2}\ \cancel{1}}$$

$$= \frac{12\ 11\ 10\ 9\ 8}{5\ 4\ 3\ 2\ 1}$$

$$= 792$$

There would be 792 different teams that can be chosen.

Example
Ted has 6 employees, three of them must be on duty during the night shift, how many ways can he choose who will work?

Does order matter? Since it does not matter, this problem can be solved by using the Fundamental Counting Principle, then dividing out the same grouping or you could use the formula for combination of 6 people being taken three at a time.

There are 6 ways to choose the first person, 5 ways to choose the second, and 4 ways to choose the third, that's 120 permutations. Each group of three employees can be ordered 3! Or 6 ways.

Handout 15.3 Page 8

So, we divide the number of permutations by the different ordering of the three employees.

$$\frac{6x5x4}{3!} = \frac{120}{6} \rightarrow 20 \text{ ways to pick the shifts}$$

By formula, we'd have $_6C_3$ = $\frac{6!}{(6-3)!x3!}$. Working that out, there would be 20 ways.

Doing these problems by hand can be very distracting, you would be able to concentrate on the problem more if you had a calculator that had permutations and combination on it. That way, when you had $_{12}C_7$, all you would do is plug those numbers in, press the appropriate buttons, and wala, you would have gotten 792. Don't you just love technology?

In summary, when order matters and there is no repetition, use a permutation. If order matters and there is repetition, then use the Fundamental Counting Principle. If order does not matter, use a combination.

In just about all cases, you can use the Fundamental Counting Principle to determine the size of the same space,

The formulas for permutation and combination just allow us to compute the answers quickly. However, if you read a problem and have trouble determining if it's a permutation or combination, then do it by the Fundamental Counting Principle.

Lesson 15 Activity 4: Circle Graphs	Time: 10-15 Minutes

1. Ask students what they know about circle graphs: when are they used? What type of information is depicted?
2. Draw a circle on the board and cut it in half. Ask students what each side represents. They will say half, 0.5, 50%
3. Cut the circle in quarters. Ask what each section represents now. (25%, ¼, 0.25)
4. Explain that circles are often used to represent parts of a whole. The sections of the graph represent a fast visual way to see information.
5. Do **pages 36-37 of the student book** together.

Lesson 15 Activity 5: Word Problems	Time: 15-20 Minutes

1) Do the word problems in the **workbook pages 50-53.**
2) Review any challenging questions.
3) If there is extra time or for students who finish quickly, change some of the percentages in the problems to decimals and fractions and reduce the fractions.

Lesson 15 Activity 6: Application	Time: 20 Minutes

1. Tell students they will draw their own circle graph estimating how they spend a typical day. Brainstorm how many times you will divide the circle, what each section represents, etc.

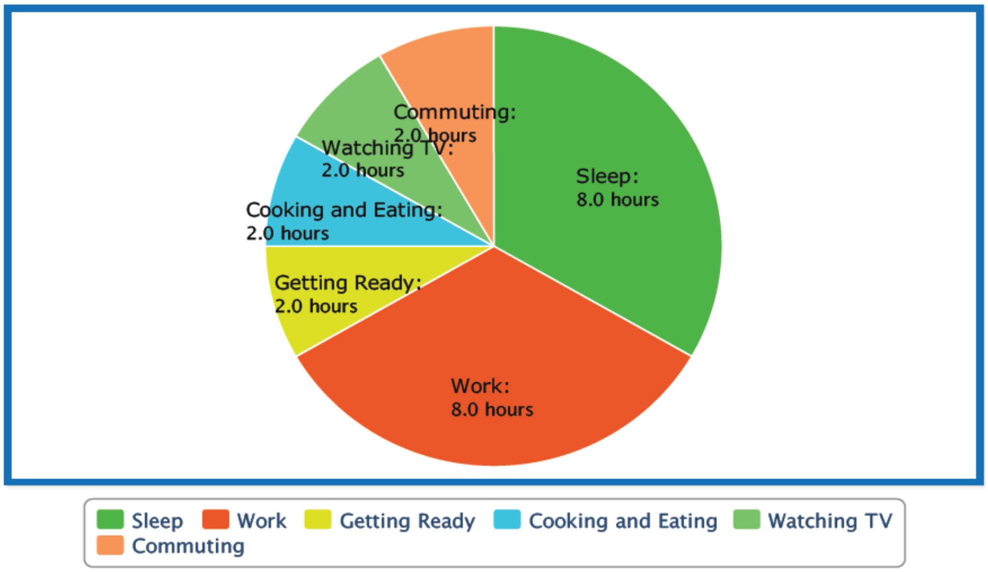
2. Have students help you when you draw your own on the board as an example. First write on the board how many hours you spend working, volunteering, commuting, cooking and eating, cleaning, reading, etc. Make sure the hours add up to 24 hours. Now they can help you decide how to divide your circle.

3. Hand out the paper, rulers, and markers.

4. For each section, write: activity, hours, fraction, and percent (if you have time). Make sure the fractions all add up to a whole.

5. The example below shows 8 hours of sleep (1/3 of day), 8 hours of work (1/3 of day), 2 hours of getting ready (1/12 of day), 2 hours of cooking and eating (1/12 of day), 2 hours of watching TV (1/12) and 2 hours of commuting (1/12).

How I Spend My Day



Lesson 15 Exit Ticket	Time: 5 Minutes

Have students work in pairs to draw circle graphs of each other's prep time on each of the 4 GED subjects (Social Studies, Science, Math and RLA)—ask them to take into account the time they spend on each subject in class, doing homework, working on online programs, taking practice tests and any other work they do outside of class. Do they spend an equal 25% of their time working on each of the four subjects, or are the proportions greater for some subjects than others? Partners will draw circle graphs of one another's reported percentages. Share results with the whole group.

Lesson 15 Extra Word Problem: Hanging Pictures	Time: 5-10 Minutes

Write on the board: Sara has 4 photos she wants to hang on the wall. Each photo is 5 inches wide and she wants to leave 2 inches in between each one.

Basic Questions:

- What is the width of all the photos from the first one to the last one if you include the space in between each one? Give your answer in feet. Draw a picture if necessary.
  - (5+2+5+2+5+2+5= 26 inches= 2 ft. 2 in. = 2 and 1/6 ft.)
- How much space is left on the 2 ends if the wall measures 16 feet 10 inches wide? Give answer in feet.
  - 16 ft.10 in. minus 2 ft. 2 in. = 14 ft. 8 in. = 14 2/3 ft.

Extension Question:

- How many different ways (in how many different orders) can Sara hang the pictures?
  - 4! = 4 x 3 x 2 x 1= 24 ways. There are 4 pictures for the first position, 3 for the second position, 2 for the third position, and only one left for the last position.